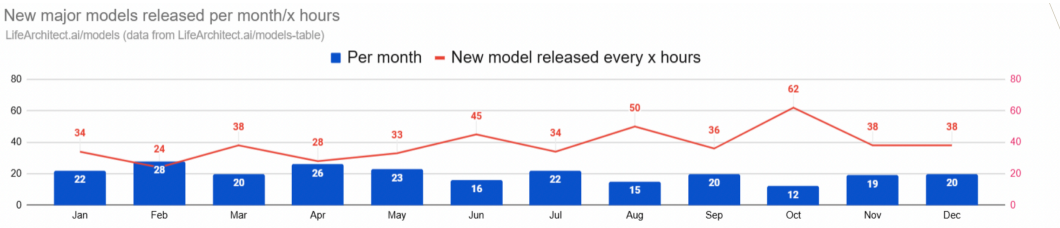
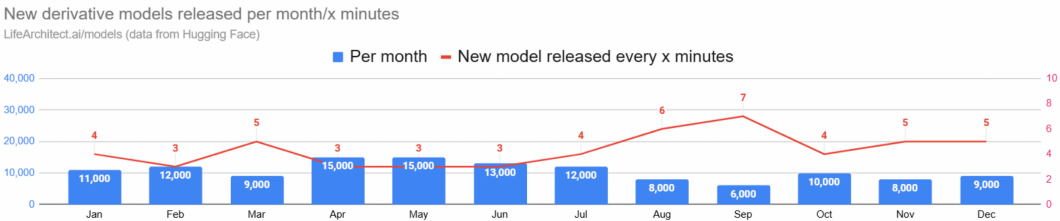
用 U 统计量揭开大模型推理的神秘面纱

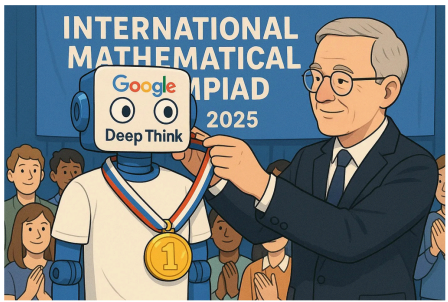
Chengchun Shi
Associate Professor of Data Science
London School of Economics and Political Science

Number of LLMs Per Month (2024)





The Year of 2025



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EDITORIAL | 17 September 2025

Bring us your LLMs: why peer review is good for AI models

Deepseek's R1 model has been peer reviewed. Others should follow the firm's example.

DeepSeekMath: Pushing the Limits of Mathematical Reasoning in Open Language Models

Zhihong Shao[1,2*†], Peiyi Wang[1,3*†], Qihao Zhu[1,3*†], Runxin Xu[1], Junxiao Song[1]
Xiao Bi[1], Haowei Zhang[1], Mingchuan Zhang[1], Y.K. Li[1], Y. Wu[1], Daya Guo[1*]

[1]DeepSeek-AI, [2]Tsinghua University, [3]Peking University

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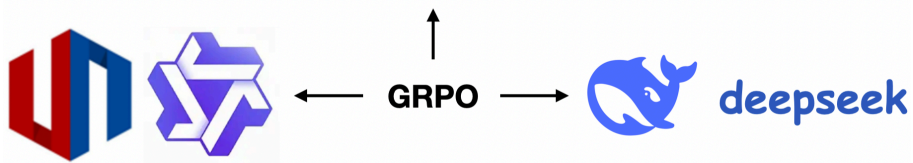
Article | Open access | Published: 17 September 2025

DeepSeek-R1 incentivizes reasoning in LLMs through reinforcement learning

Daya Guo, Dejian Yang, Haowei Zhang, Junxiao Song, Peiyi Wang, Qihao Zhu, Runxin Xu, Ruoyu Zhang, Shirong Ma, Xiao Bi, Xiaokang Zhang, Xingkai Yu, Yu Wu, Z. F. Wu, Zhibin Gou, Zhihong Shao, Zhuoshu Li, Ziyi Gao, Aixin Liu, Bing Xue, Bingxuan Wang, Bochao Wu, Bei Feng, Chengda Lu, ... Zhen Zhang + Show authors

Nature **645**, 633–638 (2025) | Cite this article



Since then, 100+ xPO variants have been proposed

Rollout Selection & Bias Correction:

- **SRPO** (Zhang et al., 2025c): history resampling
- **DAPO** (Yu et al., 2025): dynamic sampling
- **Dr.GRPO** (Liu et al., 2025): mitigates length bias
- **OPO** (Hao et al., 2025): optimal baseline to reduce gradient variance

Reward Shaping & Advantage Estimation:

- **EMPO** (Zhang et al., 2025b): fully unsupervised / minimizes predictive entropy
- **AAPO** (Xiong et al., 2025a): advantage momentum
- **BNPO** (Xiao et al., 2025): adaptively normalizes rewards via Beta distribution
- **SEED-GRPO** (Chen et al., 2025): ： uses semantic entropy to scale updates by uncertainty
- **GRPO-lead** (Zhang & Zuo, 2025): length-dependent accuracy, explicit penalties, difficulty-aware reweighting

Efficiency-Driven Methods:

- **CPPO** (Lin et al., 2025): pruning low-advantage completions
- **S-GRPO** (Dai et al., 2025b): early exit to cut redundancy
- **Ada-GRPO** (Wu et al., 2025): adaptive reasoning formats
- **GVPO** (Zhang et al., 2025a): analytical KL-constrained weighting
- **GRPO-λ** (Dai et al., 2025a): ddynamic switch between length-penalized vs length-agnostic rewards to avoid collapse

Gap between theory and practice

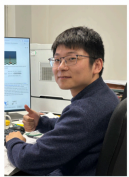
Q1 *Why is GRPO so effective?*

Q2 *What is the rationale for using the group mean to approximate the value function?*

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

Q4 *What is the optimal group size?*

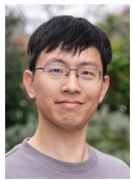
Demystifying GRPO: Its Policy Gradient is a U-Statistic



Hongyi Zhou
Tsinghua

Kai Ye
LSE

Erhan Xu
LSE



Jin Zhu
Birmingham

Ying Yang
Tsinghua

Shijin Gong
USTC

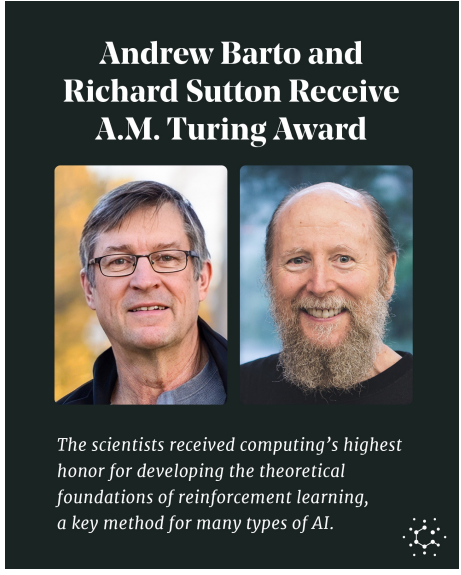
Background

1. **Reinforcement Learning**
2. **LLM Reasoning**
3. **U-statistics**

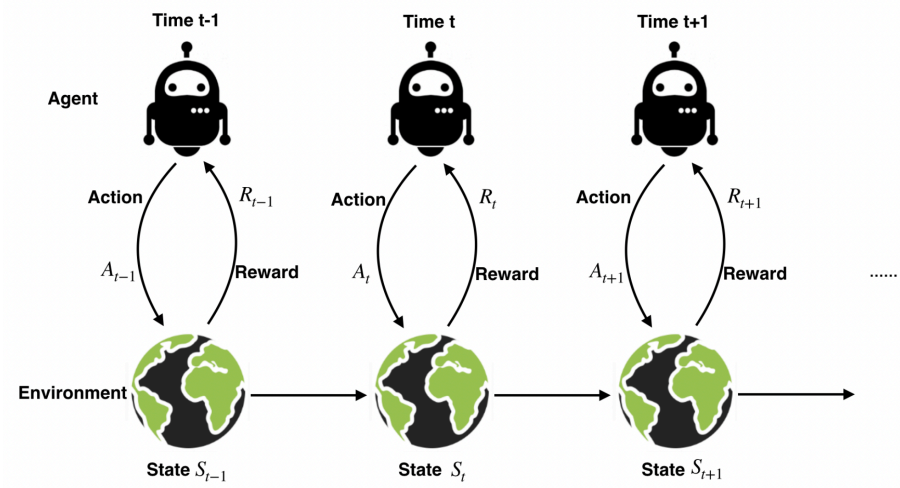
Background

1. **Reinforcement Learning**
2. **LLM Reasoning**
3. **U-statistics**

Reinforcement Learning (RL)

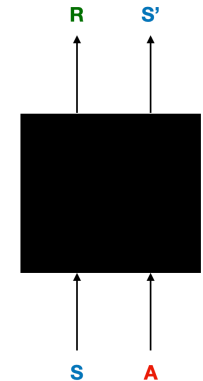


Reinforcement Learning (Cont'd)

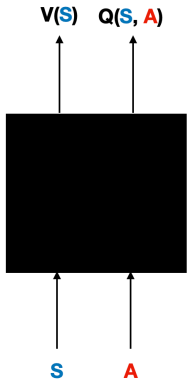


Objective: find an optimal policy that maximizes the cumulative reward

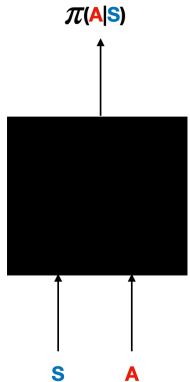
Three Types of RL Algorithms



Model-based
(Dynamic Programming)

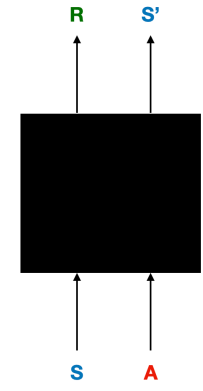


Value-based
(Q-learning)

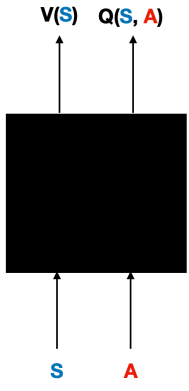


Policy-based
(Reinforce/Actor-Critic)

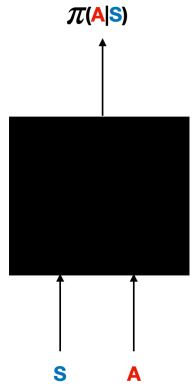
Three Types of RL Algorithms



Model-based
(Dynamic Programming)



Value-based
(Q-learning)



Policy-based
(Reinforce/Actor-Critic)

REINFORCE: Pseudocode

- **Initialization**: θ arbitrary
- **For** $i = 1, 2, \cdots, n$ **do**:

  Generate an episode $(S_0, A_0, R_0, \cdots, S_T, A_T, R_T)$ using policy π_θ

  For $t = 0, 1, 2, \cdots, T$ **do**:

$$\theta \leftarrow \theta + \eta \nabla_\theta \log(\pi_\theta(A_t|S_t)) G_t$$

  end for

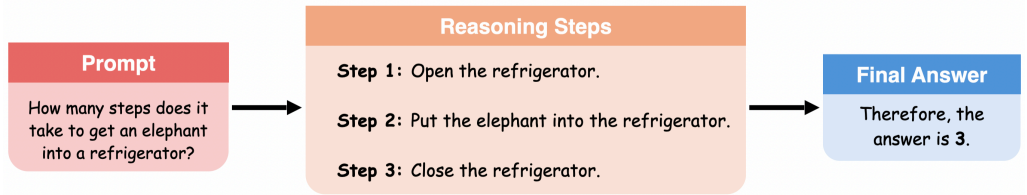
end for

return θ

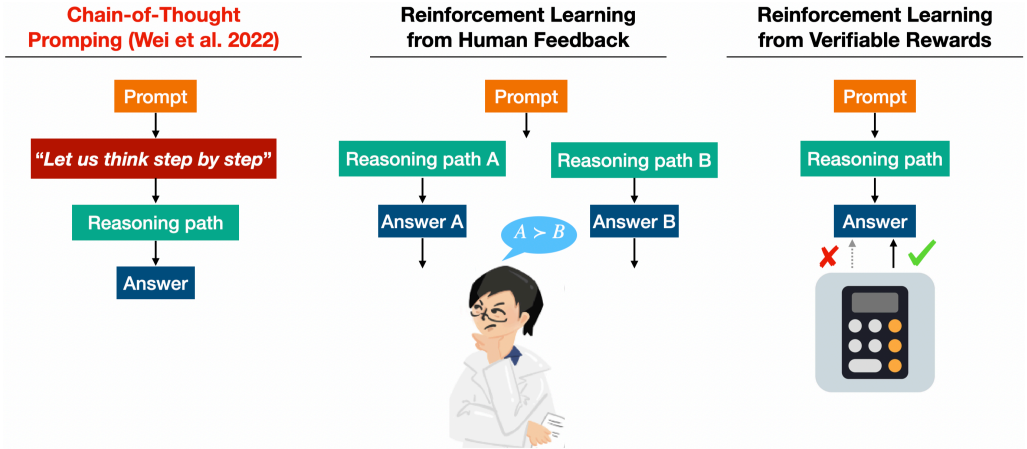
Background

1. Reinforcement Learning
2. **LLM Reasoning**
3. U-statistics

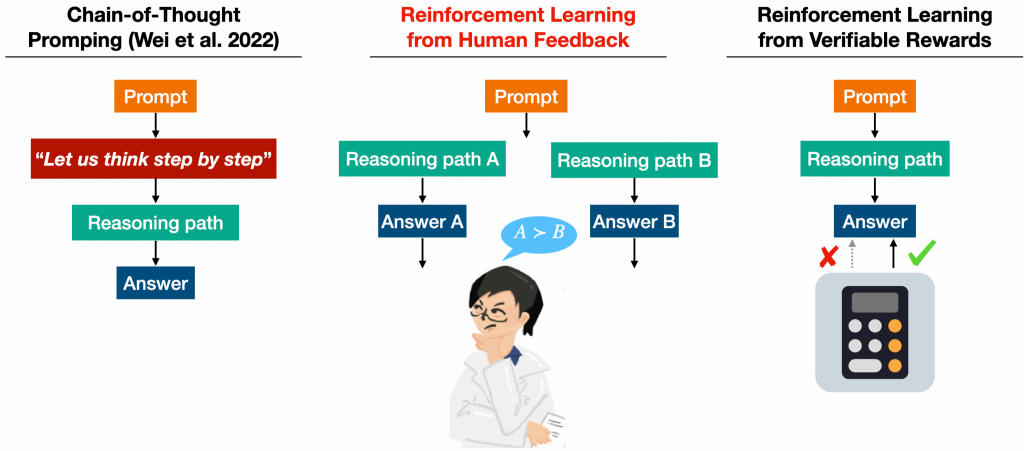
LLM Reasoning



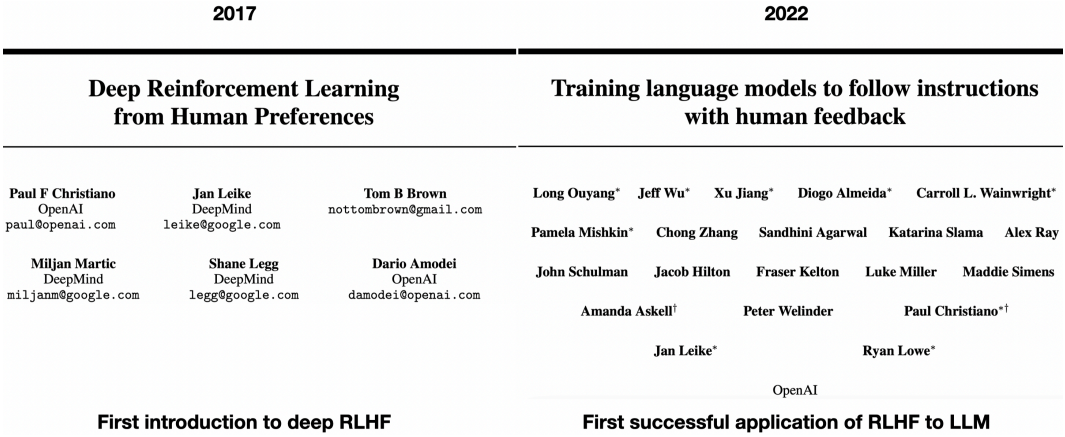
Approach I: Chain-of-Thought Prompting



Approach II: RLHF



RLHF (Cont'd)



2017

Deep Reinforcement Learning from Human Preferences

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First introduction to deep RLHF

2022

Training language models to follow instructions with human feedback

Long Ouyang* **Jeff Wu*** **Xu Jiang*** **Diogo Almeida*** **Carroll L. Wainwright***

Pamela Mishkin* **Chong Zhang** **Sandhini Agarwal** **Katarina Slama** **Alex Ray**

John Schulman **Jacob Hilton** **Fraser Kelton** **Luke Miller** **Maddie Simens**

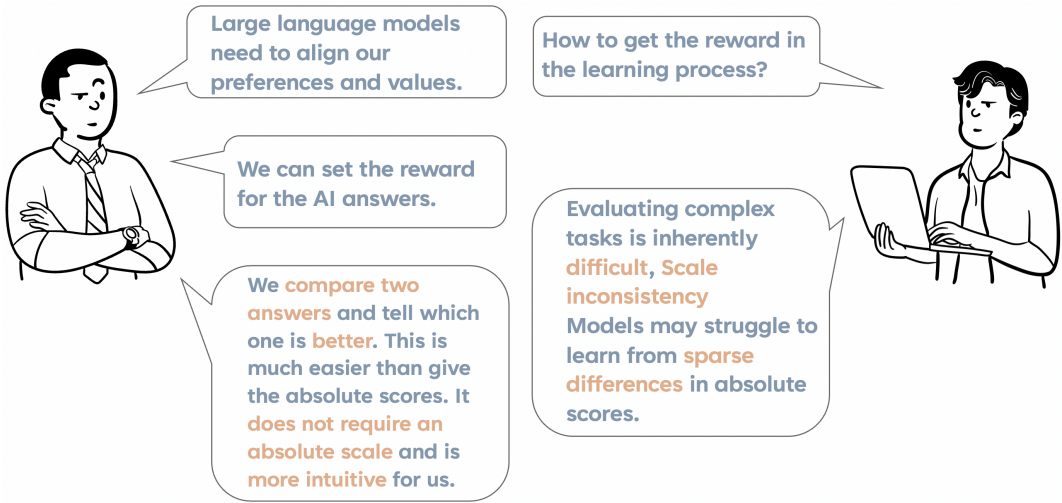
Amanda Askell† **Peter Welinder** **Paul Christiano*†**

Jan Leike* **Ryan Lowe***

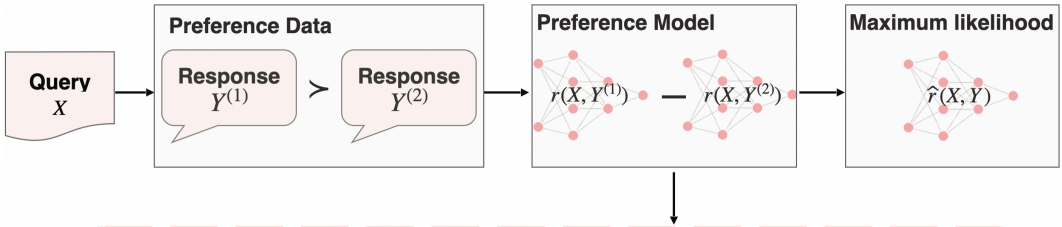
OpenAI

First successful application of RLHF to LLM

Reward learning in RLHF



Reward learning in RLHF (Cont'd)



Bradley-Terry (BT) model (Bradley & Terry, 1952) is most widely adopted to model human preferences:

$$p(Y^{(1)} \succ Y^{(2)} | X) = \sigma(r(X, Y^{(1)}) - r(X, Y^{(2)}))$$

Proximal policy optimization (Ouyang et al., 2022)

1) Policy Model π_θ

The LLM we update (the "actor")

2) Reward Model $r(x, y)$

Learn from human feedback

3) Value Function $V^{\pi_\theta}(x)$

Baseline we use to calculate the advantage function for variance reduction

4) Reference Policy π_{ref}

To prevent reward hacking

$$r(x, y) - \beta \mathrm{KL}(\pi_\theta(\bullet \mid x) \| \pi_{ref}(\bullet \mid x))$$

```
def rlhf_training(policy_model,reward_model,value_model,prompts):
    ref_policy = freeze(copy(policy_model))
    for step in range(num_steps):
        experiences = []
        for prompt in prompts:
            # Generate response
            response = policy_model.generate(prompt)
            # Get reward from reward model
            reward = reward_model(prompt, response)
            logp_old = policy_model.logprob(prompt, response)
            # Add KL penalty (stay close to reference policy)
            kl_penalty = kl_divergence(policy_model, ref_policy)
            final_reward = reward - beta * kl_penalty
            experiences.append((prompt, response,
            final_reward, logp_old))
            # Advantage (usually estimated by GAE)
            adv = compute_advantages(experiences, value_model)
            experience.append(adv)
        # PPO update
        for _ in range(ppo_epochs):
            ppo_step(policy_model, experiences)
    return policy_model
```

PPO (Cont'd)

1) Policy Model π_θ

The LLM we update (the "actor")

2) Reward Model $r(x, y)$

Challenge I: Human labelling is costly in reasoning

3) Value Function $V^{\pi_\theta}(x)$

Challenge II: Estimating and storing a value function is computationally inefficient

4) Reference Policy π_{ref}

To prevent reward hacking

$$r(x, y) - \beta \mathrm{KL}(\pi_\theta(\bullet|x) \| \pi_{ref}(\bullet|x))$$

```python
def rlhf_training(policy_model,reward_model,value_model,prompts):
    ref_policy = freeze(copy(policy_model))
    for step in range(num_steps):
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            adv = compute_advantages(experiences, value_model)
            experience.append(adv)
        # PPO update
        for _ in range(ppo_epochs):
            ppo_step(policy_model, experiences)
    return policy_model
```
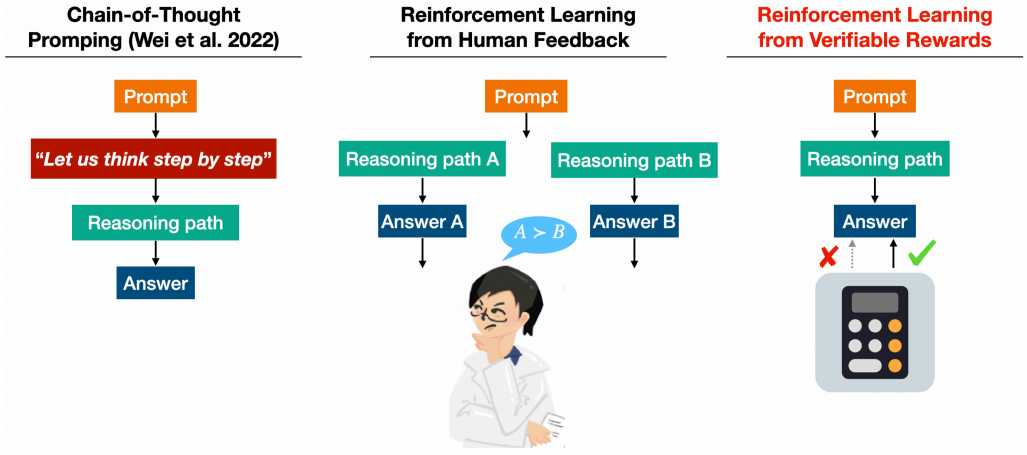

Approach III: RLVR



RLVR (Lambert et al., 2024)

Motivation

Problem with RLHF:

- Human preferences are subjective
- Reward model can be inaccurate
- Hard to verify correctness

RLVR Solution:

- Use objective, verifiable rewards
- Automatic verification
- Ground truth available
- Scalable and reliable

Key Principle

Instead of "Is this good?" ask "Is this correct?"

Suitable for:

- **Code**: Run unit tests ✓
- **Math**: Check answer ✓
- **Logic**: Verify proof ✓

$$R_{\text{RLVR}}(x, y) = \begin{cases} +1 & \text{if correct} \\ 0 & \text{if wrong} \end{cases}$$

From REINFORCE to RLVR

- **Initialization**: θ arbitrary
- **For** $i = 1, 2, \cdots, n$ **do**:

  Sample an prompt X

  Generate a completion Y (reasoning trace + response) using policy π_θ

  Obtain the verifiable reward $Z = r(X, Y)$:

$$\theta \leftarrow \theta + \eta \nabla_\theta \log(\pi_\theta(Y|X))Z$$

end for

return θ

Variance reduction using a baseline

- **Initialization**: θ arbitrary
- **For** $i = 1, 2, \cdots, n$ **do**:

  Sample an prompt X

  Generate a completion Y using policy π_θ

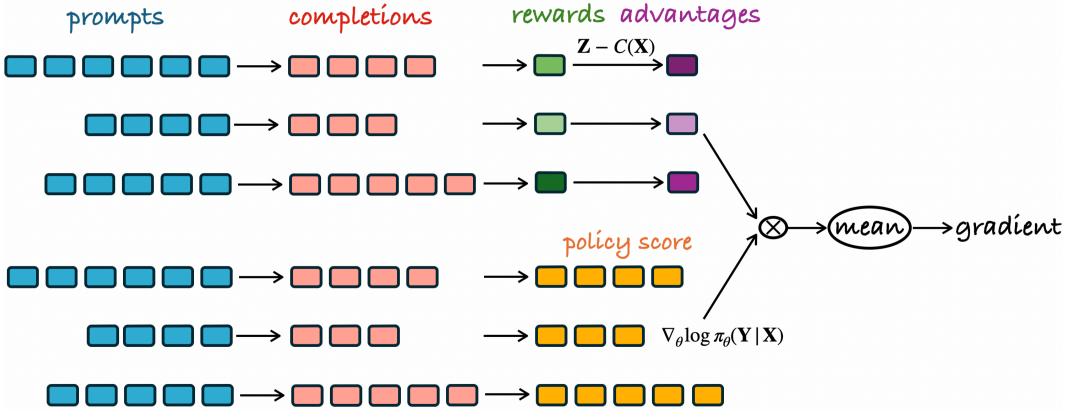
  Obtain the verifiable reward $Z = r(X, Y)$:

$$\theta \leftarrow \theta + \eta \nabla_\theta \log(\pi_\theta(Y|X))[Z - C(X)]$$

  end for

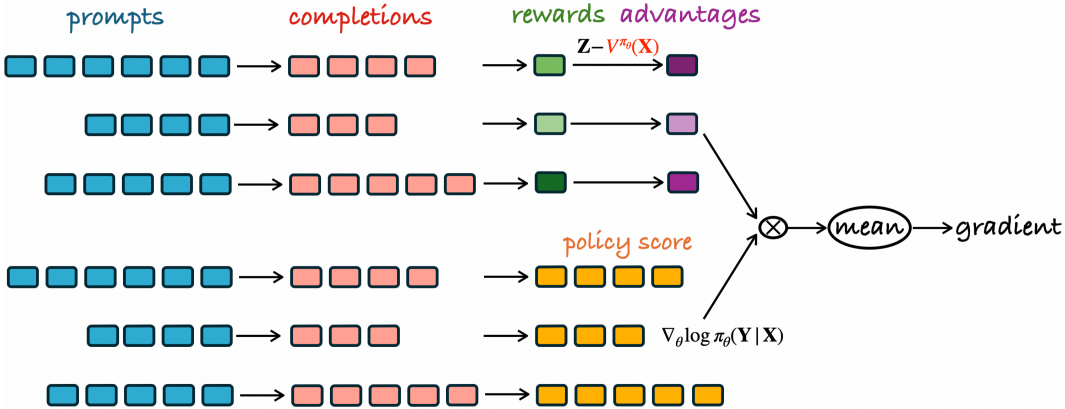
  return θ

A minibatch version



$$\theta \leftarrow \theta + \eta \text{ gradient}$$

An oracle version



$$\theta \leftarrow \theta + \eta \text{ gradient}$$

Background

1. Reinforcement Learning
2. LLM Reasoning
3. **U-statistics**

Second-order U-statistics

- $\{\boldsymbol{X_i}\}_{\boldsymbol{i=1}}^{\boldsymbol{n}}$ denotes a sequence of i.i.d. random vectors
- Given a symmetric kernel function $\boldsymbol{h}$
- A second-order U-statistic is defined as

$$\boldsymbol{U} = \frac{\mathbf{1}}{\boldsymbol{n}(\boldsymbol{n}-\mathbf{1})} \sum_{\boldsymbol{i}\neq\boldsymbol{j}} \boldsymbol{h}(\boldsymbol{X_i}, \boldsymbol{X_j}).$$

- Hoeffding decomposition:

$$\boldsymbol{U} = \boldsymbol{h}_0 + \underbrace{\frac{\mathbf{2}}{\boldsymbol{n}} \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} [\boldsymbol{h}_1(\boldsymbol{X_i}) - \boldsymbol{h}_0]}_{\text{first-order term } O_p(\boldsymbol{n}^{-1/2})} + \underbrace{\frac{\mathbf{1}}{\boldsymbol{n}(\boldsymbol{n}-\mathbf{1})} \sum_{\boldsymbol{i}\neq\boldsymbol{j}} [\boldsymbol{h}(\boldsymbol{X_i}, \boldsymbol{X_j}) - \boldsymbol{h}_1(\boldsymbol{X_i}) - \boldsymbol{h}_1(\boldsymbol{X_j}) + \boldsymbol{h}_0]}_{\text{second-order term } O_p(\boldsymbol{n}^{-1})},$$

where $\boldsymbol{h}_0 = \mathbb{E}[\boldsymbol{h}(\boldsymbol{X_1}, \boldsymbol{X_2})]$ is the expectation of the kernel and $\boldsymbol{h}_1(\boldsymbol{x}) = \mathbb{E}[\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{X_2})]$ denotes the first-order projection.

Bring it all together: GRPO

Motivation

Problem with PPO:

- a separate value model/critic increases GPU memory.
- a poorly learned value model makes advantages noisy, destabilizing updates.

GRPO Solution:

- No value network needed
- Uses group-based advantages

Key Idea

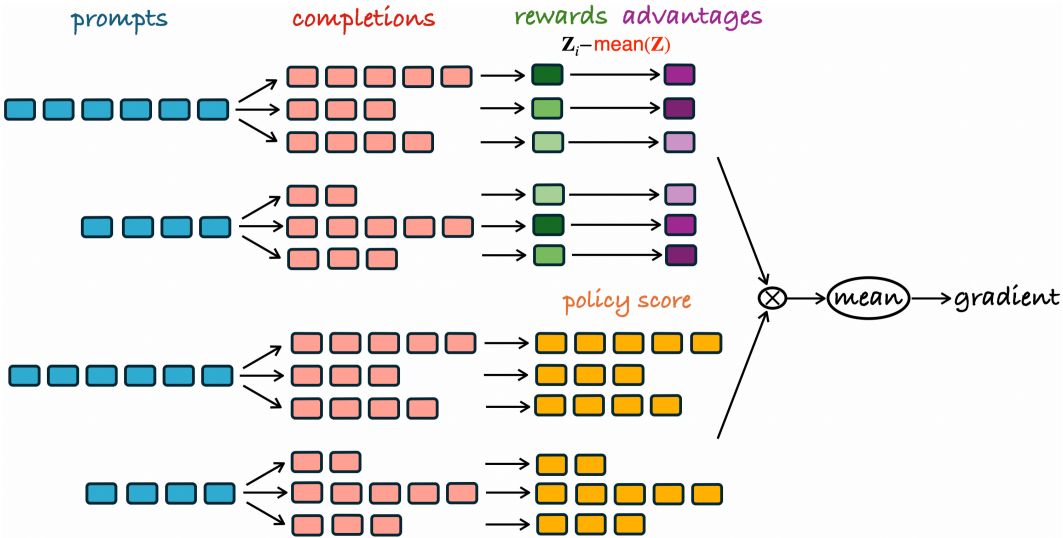
Instead of comparing to a baseline **by model**, compare responses **within a group**:

1. Generate **multiple responses** for same prompt
2. Score all responses with reward model
3. Use **group statistics** for advantage
4. Update policy based on relative quality

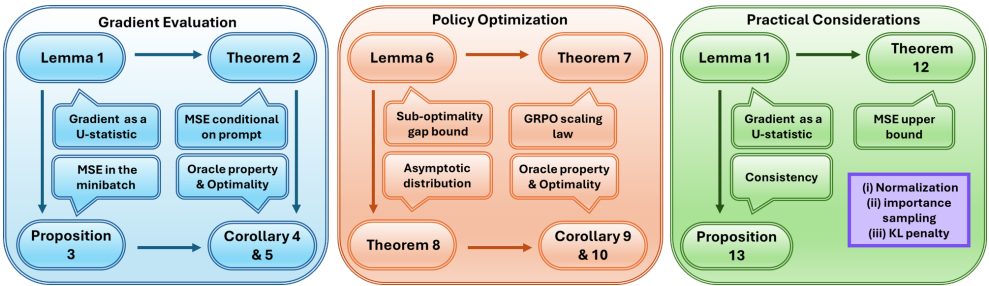
$$\text{Advantage}_i = \mathbf{Z}_i - \text{mean}(\mathbf{Z})$$

No separate value network needed!

GRPO-type algorithm



Demystify GRPO: Our results



Demystify GRPO (Cont'd)

Q1 *Why is GRPO so effective?*

Q2 *What is the rationale for using the group mean to approximate the value function?*

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

Q4 *What is the optimal group size?*

GRPO's gradient is a U-statistic

- $\boldsymbol{G} =$ no. of sampled completions per prompt.
- GRPO-type gradient:

$$\widehat{g}_{\mathrm{GRPO}}(\boldsymbol{x};\boldsymbol{\theta}) = \frac{1}{G-1}\sum_{g=1}^{G} \underbrace{\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{Y}^{(g)} \mid \boldsymbol{x})}_{\text{policy score}} \underbrace{[\boldsymbol{Z}^{(g)} - \mathbf{mean}(\boldsymbol{Z})]}_{\text{advantage}}.$$

引理

$\widehat{g}_{\mathrm{GRPO}}(\boldsymbol{x};\boldsymbol{\theta})$ *can be written as a second-order U-statistic.*

Proof

- Each individual term in $\widehat{g}_{\mathrm{GRPO}}(\boldsymbol{x};\boldsymbol{\theta})$ equals

$$\frac{G}{G-1}\nabla_\theta \log \pi_\theta(\boldsymbol{Y}^{(g)} \mid \boldsymbol{x})[\boldsymbol{Z}^{(g)} - \mathbf{mean}(\boldsymbol{Z})] = \frac{1}{G-1}\sum_{k\neq g} \nabla_\theta \log \pi_\theta(\boldsymbol{Y}^{(g)} \mid \boldsymbol{x})[\boldsymbol{Z}^{(g)} - \boldsymbol{Z}^{(k)}]$$

- **Averaging** over all $g \in \{1,\ldots,G\}$ & applying **symmetrization**

$$\widehat{g}_{\mathrm{GRPO}}(\boldsymbol{x};\boldsymbol{\theta}) = \binom{G}{2}^{-1} \sum_{1\leq i<j\leq G} \frac{1}{2}[\nabla_\theta \log \pi_\theta(\boldsymbol{Y}^{(i)}|\boldsymbol{x}) - \nabla_\theta \log \pi_\theta(\boldsymbol{Y}^{(j)}|\boldsymbol{x})](\boldsymbol{Z}^{(i)} - \boldsymbol{Z}^{(j)})$$

According to Hoeffding decomposition

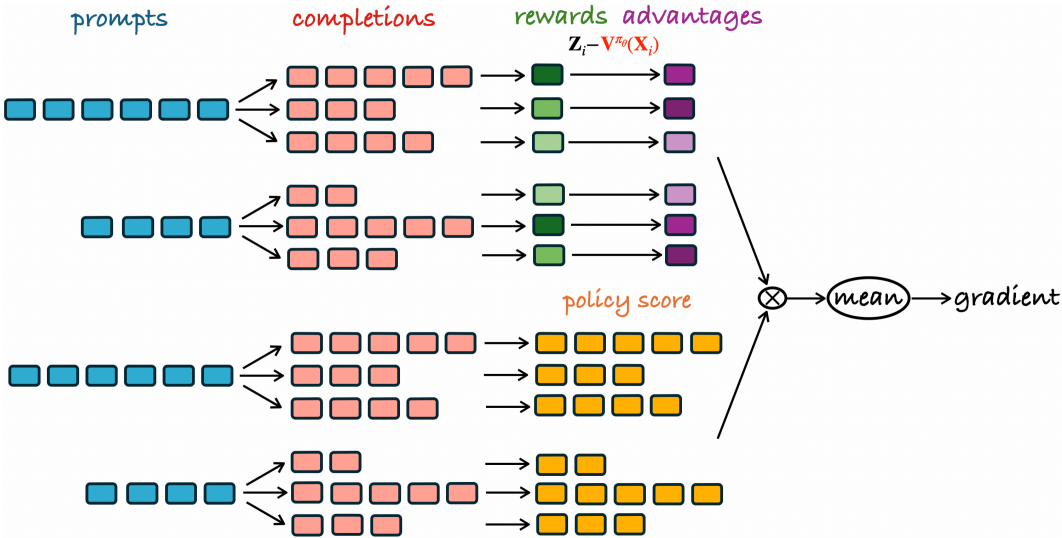
$$\widehat{g}_{\mathrm{GRPO}}(\boldsymbol{x};\boldsymbol{\theta}) = \text{true gradient at } \boldsymbol{x} + \underbrace{\widehat{g}_{\mathrm{oracle}}(\boldsymbol{x};\boldsymbol{\theta}) - \mathbb{E}[\widehat{g}_{\mathrm{oracle}}(\boldsymbol{x};\boldsymbol{\theta})]}_{\text{first-orderterm } O_p(\boldsymbol{G}^{-1/2})} + \underbrace{\text{second-order term}}_{O_p(\boldsymbol{G}^{-1})}$$

- $\boldsymbol{B}$ denotes the number of prompts
- Consider the minibatch version $\widehat{g}_{\mathrm{GRPO}}(\boldsymbol{\theta}) = \frac{1}{B}\sum_{b=1}^{B} \widehat{g}_{\mathrm{GRPO}}(\boldsymbol{X}^{(b)};\boldsymbol{\theta})$

定理

$$MSE(\widehat{g}_{GRPO}(\boldsymbol{\theta})) = MSE(\widehat{g}_{oracle}(\boldsymbol{\theta})) + O\Big(\frac{\mathbb{E}\|\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{Y}|\boldsymbol{X})\|^2}{BG^2}\Big)$$

Oracle algorithm



Demystify GRPO (Cont'd)

Q1 *Why is GRPO so effective?*

Q2 *What is the rationale for using the group mean to approximate the value function?*

A2 Use of group mean provides a **first-order approximation** to the gradient of the **oracle algorithm** with access to the value function

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

Q4 *What is the optimal group size?*

Demystify GRPO (Cont'd)

Q1 *Why is GRPO so effective?*

Q2 *What is the rationale for using the group mean to approximate the value function?*

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

Q4 *What is the optimal group size?*

Demystify GRPO (Cont'd)

Define the **suboptimality gap** of a policy π

$$\sup_{\theta} \mathbb{E}^{\pi_\theta}(Z) - \mathbb{E}^{\pi}(Z)$$

推论 (Oracle property)

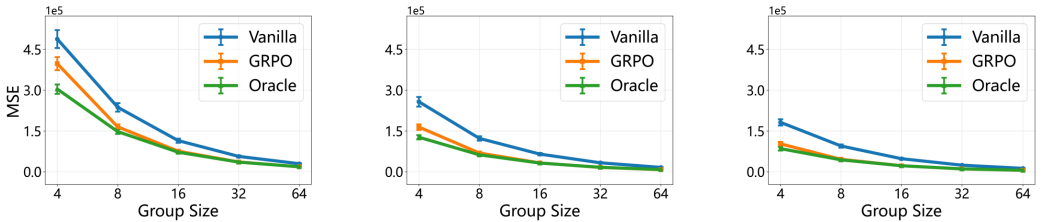
*As $G \to \infty$, both (i) the MSE of the GRPO-type gradient and (ii) the suboptimality gap of its induced policy are asymptotically equivalent to those of the **oracle** algorithm.*

推论 (Optimality)

*As $G \to \infty$, both (i) the MSE of the GRPO-type gradient and (ii) the suboptimality gap of its induced policy are asymptotically no larger than those of **any** algorithm using a generic baseline $C(X)$. In particular, the gradient MSE is strictly smaller than that of the **vanilla** algorithm with zero baseline, under mild conditions.*

In DeepSeekMath, $G = 64$, which is sufficiently large to approximate the asymptotics.

Experiment I: Gradient evaluation



(a) Base model *(b) Instruct model* *(c) ICL model*

Figure 4: MSEs of three policy gradient estimators (vanilla, GRPO-type, and oracle) under three model configurations (base, instruct and in-context learning (ICL)) for different group sizes. Error bars represent 95% confidence intervals of the empirically estimated MSEs.

Demystify GRPO (Cont'd)

Q1 *Why is GRPO so effective?*

A1 Because it achieves **oracle** performance without estimating the value function and is asymptotically **optimal** in both **MSE** and **suboptimality gap**.

Q2 *What is the rationale for using the group mean to approximate the value function?*

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

Q4 *What is the optimal group size?*

Demystify GRPO (Cont'd)

Q1 *Why is GRPO so effective?*

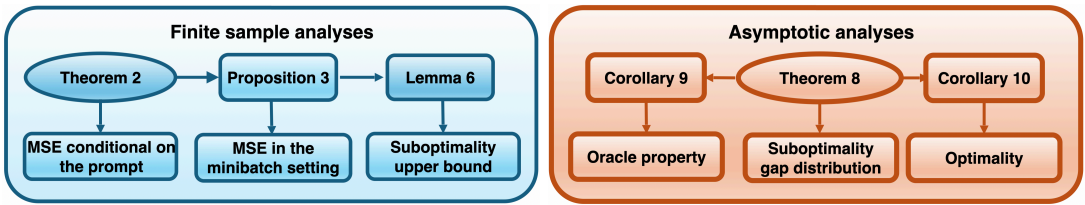
Q2 *What is the rationale for using the group mean to approximate the value function?*

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

A3 Yes – this is exactly what we establish in the paper.

Q4 *What is the optimal group size?*

Finite-sample & asymptotic analyses



Our asymptotic analyses are novel in two aspects:

1. **ML:** focuses on error bounds for the suboptimality gap, capturing only its **order** (not its distribution)
2. **Statistics:** establishes asymptotic results under **parameter identifiability** (clearly violated in overparameterized LLMs), which we do not require

Demystify GRPO (Cont'd)

Q1 *Why is GRPO so effective?*

Q2 *What is the rationale for using the group mean to approximate the value function?*

Q3 *Can we provide finite-sample or asymptotic analyses regarding its convergence?*

Q4 *What is the optimal group size?*

A4 We derive a **scaling law**

(i) delineates how GRPO's performance depends on the group size

(ii) identifies the optimal group size that maximizes its performance

GRPO scaling law

定理

GRPO's sub-optimality upper bound depend on (i) the number of sampled prompts B and (ii) the number of sampled completions G per prompt through the following:

$$\frac{c_1}{B} + \frac{c_2}{BG} + \frac{c_3}{BG^2},$$

*where c_1, c_2, c_3 depend only on the **data generating process** and **geometry of the policy space**.*
Under a fixed sampling budget $N = BG$ per iteration or a total sampling budget $\boldsymbol{N} = nBG$, the optimal group size G^* that minimizes the upper bound is:

$$G^* = \sqrt{\frac{c_3}{c_1}}.$$

Universality of G^*

The optimal group size G^* is **universal**:

- independent of the **budget** N
- independent of the **number of iterations** n
- independent of the **learning rate schedule**

depending solely on the **data generating process** and **geometry of the policy space**.

Experiment II: Universality across training iterations

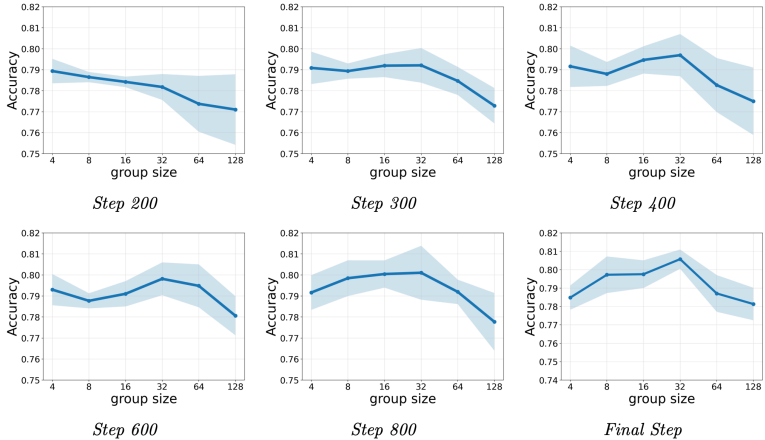


Figure 5: Test accuracy of GRPO-fine-tuned models at different training steps with a fixed sampling budget of $N = B \times G = 1024$ per prompt. Both training and evaluation are conducted on GSM8K. Each curve shows accuracy as a function of the group size $G \in \{4, 8, 16, 32, 64, 128\}$. Results are averaged over five independent runs, with shaded regions visualizing 95% confidence bands.

Experiment III: Universality across sampling budgets

Table 2: Test accuracy of GRPO-fine-tuned models at the final training step. Each row reports accuracy as a function of the group size $G \in 4, 8, 16, 32, 64, 128$ with a fixed sampling budget per prompt; the sampling budget varies across rows. Both training and evaluation are conducted on MATH. Due to the high computational cost of training a 7B model, results are reported from a single run, with the highest accuracy highlighted in bold.

Sampling budget	Group size G					
	4	8	16	32	64	128
1024	0.7677	0.7491	0.7753	0.7627	**0.7817**	0.7743
2048	0.7703	0.7679	0.7697	0.7793	**0.7819**	0.7665
4096	0.7691	0.7713	0.7701	0.7673	0.7703	**0.7757**

Thank You!

☺Papers can be found on my personal website

`callmespring.github.io`